

# Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point

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**ABSTRACT:** We extend the definition of “spectral dimension” (usually defined for fractal and lattice geometries) to theories on smooth spacetimes with anisotropic scaling. We show that in quantum gravity dominated by a Lifshitz point with dynamical critical exponent  $z$  in  $D+1$  spacetime dimensions, the spectral dimension of spacetime is equal to

$$d_s = 1 + \frac{D}{z}.$$

In the case of gravity in  $3+1$  dimensions presented in arXiv:0901.3775, which is dominated by  $z=3$  in the UV and flows to  $z=1$  in the IR, the spectral dimension of spacetime flows from  $d_s=4$  at large scales, to  $d_s=2$  at short distances. Remarkably, this is the qualitative behavior of  $d_s$  found numerically by Ambjørn, Jurkiewicz and Loll in their *causal dynamical triangulations* approach to quantum gravity.

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## 1. Introduction

The idea that the effective spacetime dimension can change with the scale is not new.

One simple thing that can happen as we probe spacetime at shorter distances is that extra dimensions can emerge. The fact that our macroscopic Universe appears, to a good approximation, four-dimensional is then viewed as a result of course graining. Such extra dimensions can be of the Kaluza-Klein type [1–3], or our observed universe can be the boundary of a higher-dimensional space [4–7], or a higher codimension brane, perhaps with additional warping of the full geometry.

Another intriguing possibility is that the nature of the four macroscopic spacetime dimensions themselves may qualitatively change with the changing scale. The poor short-distance behavior of general relativity has often been interpreted as an indication that something radical must happen to spacetime at short distances. It has been speculated that at some characteristic scale (often related to the Planck scale), the smooth geometry of spacetime could be replaced by a discrete structure, or exhibit some form of fractal behavior, or a stringy generalization of geometry, or that the short-distance nature of spacetime might be non-geometric altogether. This picture is further supported by our current understanding of string theory, in which the macroscopic spacetime – or at least space – can often be viewed as an emergent concept.

In recent numerical simulations of lattice quantum gravity in the framework of causal dynamical triangulations (CDT) [8–10], an interesting phenomenon has been observed: The system exhibits a phase in which the effective spacetime dimension is four at large scales, but changes continuously to two at short distances [11]. The four-dimensional nature of spacetime at large scales indicates that the model does exhibit a good long-distance continuum limit. However, the interpretation of the effective change in dimension at shorter scales is not clear.

Perhaps the geometry undergoes a dynamical dimensional reduction, or develops a foamy structure at short distances. The lattice methods of dynamical triangulations do not offer enough analytical control over the details of the dynamics of geometry, and it would be desirable to compare the findings against an analytical tool in a continuum framework.

The CDT approach to quantum gravity has one distinguishing feature: The triangulations are restricted to conform to a preferred causal structure, given by a preferred foliation by slices of constant time. This restriction is motivated by the desire to maintain causality and leads to the suppression of baby universes, which – when present – are believed to be responsible for the pathological branched-polymer scaling in the continuum limit.

The preferred causal structure imposed on spacetimes in the CDT framework is quite reminiscent of the symmetries in the recently proposed Lifshitz phase of quantum gravity [12,13]. This theory – defined in the path-integral framework – exhibits an anisotropic scaling of spacetime at short distances. The degree of anisotropy is measured by the dynamical critical exponent  $z$ , which changes from  $z = 3$  in the UV to the relativistic value  $z = 1$  in the IR. In this paper, we present some evidence suggesting that the CDT approach to lattice gravity may in fact be a lattice version of the quantum gravity at a Lifshitz point. Using the same definition of dimension as in the CDT approach [11], we show that in the continuum framework of [13] the effective dimension of the Universe flows from four at large distances to two at short distances, reproducing the lattice results of [11].

## 2. The Spectral Dimension of Fluctuating Geometries

In principle, there are many different ways of defining the dimension of a fluctuating geometry. Here we follow [11], and consider a measure of dimension which has proven useful in discretized approaches to quantum gravity in low dimensions: the “spectral dimension” of spacetime. The idea is simple: Spectral dimension of a geometric object  $\mathcal{M}$  is the effective dimension of  $\mathcal{M}$  as seen by an appropriately defined diffusion process (or a random walker) on  $\mathcal{M}$ . The diffusion process is characterized by the probability density  $\rho(\mathbf{w}, \mathbf{w}'; \sigma)$  of diffusion from point  $\mathbf{w}$  to point  $\mathbf{w}'$  in  $\mathcal{M}$ , in diffusion time  $\sigma$ , subjected to the initial condition  $\rho(\mathbf{w}, \mathbf{w}'; 0) = \delta(\mathbf{w} - \mathbf{w}')$ . The *average return probability*  $P(\sigma)$  is obtained by evaluating  $\rho(\mathbf{w}, \mathbf{w}'; \sigma)$  at  $\mathbf{w} = \mathbf{w}'$  and averaging over all points  $\mathbf{w}$  in  $\mathcal{M}$ . The *spectral dimension* of  $\mathcal{M}$  is then defined in terms of  $P(\sigma)$ ,

$$d_s = -2 \frac{d \log P(\sigma)}{d \log \sigma}. \quad (2.1)$$

For example, in the case of  $\mathcal{M} = \mathbf{R}^d$  with the flat Euclidean metric, we obtain

$$\rho(\mathbf{w}, \mathbf{w}'; \sigma) = \frac{e^{-(\mathbf{w} - \mathbf{w}')^2 / 4\sigma}}{(4\pi\sigma)^{d/2}}. \quad (2.2)$$

In this case, the spectral dimension (2.1) is  $d_s(\mathbf{R}^d) = d$ , which simply reproduces the topological dimension of the Euclidean space.

The spectral dimension can be defined in a manifestly coordinate-independent way, which makes it applicable to a wide range of geometric objects beyond smooth manifolds, including those with various forms of fractal behavior. Indeed, objects are known for which  $d_s$  is not an integer: For example, the spectral dimension of branched polymers [14] is  $d_s = 4/3$ .

The spectral dimension has been used [15–22] as one of the simplest observables probing the continuum limit in the lattice approach to quantum gravity in two dimensions. This case is relevant for the description of fluctuating worldsheets in noncritical string theory. In the nonperturbative definition of the system in terms of dynamical triangulations and matrix models, the spectral dimension of worldsheets has been found to be  $d_s = 2$  [19], as long as the central charge of the worldsheet matter sector is  $c \leq 1$ . Above this  $c = 1$  barrier, the ensemble of fluctuating geometries is believed to collapse to a branched polymer phase. This expectation has been further confirmed by the measurement of the spectral dimension in [15], yielding  $d_s = 4/3$  above  $c = 1$ . Interestingly, this simplest branched polymer phase of two-dimensional gravity is in fact the lowest member of an infinite family of multi-critical phases [16], parametrized by  $m = 2, 3, \dots$ , and with spectral dimensions

$$d_s = 2m/(2m - 1). \quad (2.3)$$

In [11], the spectral dimension of spacetime was measured in the numerical CDT approach to quantum gravity in 3+1 dimensions, with intriguing results. At long distances, the spectral dimension found by [11] is

$$d_s = 4.02 \pm 0.1, \quad (2.4)$$

*i.e.*, the spacetime is macroscopically four-dimensional. With the changing scale, however, the spectral dimension appears to smoothly decrease to the short-distance limit, given by [11]

$$d_s = 1.80 \pm 0.25. \quad (2.5)$$

This value is consistent with the effective reduction of spacetime to two dimensions at short distances.

As we will show, a similar reduction in the spectral dimension of spacetime is found in the continuum path-integral approach to quantum gravity with anisotropic scaling, presented in [12, 13].

### 3. Gravity at a Lifshitz Point

The anisotropic scaling of spacetime is characterized by the dynamical critical exponent  $z$ ,

$$\mathbf{x} \rightarrow b\mathbf{x}, \quad t \rightarrow b^z t. \quad (3.1)$$

Models with anisotropic scaling are common in condensed matter (see, *e.g.*, [23]). Theories of gravity with various values of  $z$  in various spacetime dimensions  $D + 1$  were introduced in [12, 13]. The case of Yang-Mills with  $z = 2$  was discussed in [24].

For power-counting renormalizability of gravity in  $3 + 1$  dimensions, we need  $z = 3$  at short distances [13] (see also [25]). A theory of gravity in  $3 + 1$  dimensions with  $z = 3$  was presented in [13]. The field content consists of the spatial metric  $g_{ij}$ , together with the lapse and shift variables  $N_i$  and  $N$ . The theory is invariant under foliation-preserving diffeomorphisms  $\text{Diff}_{\mathcal{F}}(M)$  of spacetime, which take the coordinate form

$$\tilde{x}^i = \tilde{x}^i(t, x^j), \quad \tilde{t} = \tilde{t}(t). \quad (3.2)$$

The action is given by

$$S = \frac{2}{\kappa^2} \int dt d^3\mathbf{x} \sqrt{g} N \left\{ K_{ij} K^{ij} - \lambda (K_i^i)^2 - \mathcal{V} \right\}. \quad (3.3)$$

Here

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \quad (3.4)$$

is the extrinsic curvature tensor of the preferred time foliation  $\mathcal{F}$  of spacetime. In gravity with anisotropic scaling and  $\text{Diff}_{\mathcal{F}}(M)$  gauge symmetry,  $K_{ij}$  plays the role of the covariant time derivative of the metric tensor. The first two terms in (3.3) represent the covariant kinetic term, of second order in the time derivatives of the metric, with  $\kappa$  and  $\lambda$  two dimensionless couplings left undetermined by the gauge symmetries of  $\text{Diff}_{\mathcal{F}}(M)$ .

The potential term  $\mathcal{V}$  in (3.3) is a local function of  $g_{ij}$  and its spatial derivatives, but independent of  $\dot{g}_{ij}$ . Unlike the kinetic term quadratic in  $K_{ij}$ , which is universal and independent of the choice of  $z$ , the precise form of  $\mathcal{V}$  depends on the desired value of  $z$ . For example, general relativity requires  $\mathcal{V} \propto R - 2\Lambda$  (and  $\lambda = 1$ , to satisfy full spacetime diffeomorphism invariance), implying of course the relativistic value of  $z = 1$ .

In condensed matter, a particularly interesting class of models with  $z \neq 1$  satisfies an additional condition of “detailed balance.” Those models are intimately related to a Euclidean theory in one lower dimension. In the case of gravity in  $3+1$  dimensions, this condition means that  $\mathcal{V} \sim (\delta W / \delta g_{ij})^2$ , where  $W$  is the action of a gravity theory in three dimensions. (The square is performed with the appropriate De Witt metric; see [12, 13] for details).

The  $z = 3$  gravity introduced in [13] is described by (3.3) with

$$\mathcal{V} = \frac{\kappa^4}{16w^4} C_{ij} C^{ij}, \quad (3.5)$$

where  $C_{ij}$  is the Cotton tensor,

$$C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R_\ell^j - \frac{1}{4} \delta_\ell^j R \right), \quad (3.6)$$

and  $w$  is a dimensionless coupling constant. Since  $C_{ij} = 0$  is the equation of motion following from the variation of the Chern-Simons action  $W = (1/w^2) \int (\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma)$ , the theory satisfies detailed balance. This condition can be explicitly broken by the addition of diffeomorphism invariant terms of sixth order in spatial derivatives, such as  $\nabla_i R_{jk} \nabla^i R^{jk}$  or  $R_j^i R_k^j R_i^k$ , to the potential  $\mathcal{V}$ . This does not change the value of  $z$ , but theories without

detailed balance are generally more complex due to the proliferation of independent terms in the action. Luckily, the spectral dimension that we consider below turns out to be a very universal observable, sensitive only to the scaling (3.1) but not to the details of  $\mathcal{V}$ .

The global scaling transformations (3.1) can be generalized to the case when the spacetime geometry is fluctuating and the background is no longer flat. In [13], the local anisotropic Weyl transformations with  $z = 3$  were introduced,

$$g_{ij} \rightarrow e^{2\Omega(\mathbf{x},t)} g_{ij}, \quad N_i \rightarrow e^{2\Omega(\mathbf{x},t)} N_i, \quad N \rightarrow e^{3\Omega(\mathbf{x},t)} N. \quad (3.7)$$

(Other values of  $z$  were discussed in [12].) These represent a local version of the global anisotropic scaling (3.1) of flat space, adapted to the general background  $g_{ij}$ ,  $N_i$  and  $N$ . The anisotropic Weyl transformations (3.7) form a closed symmetry group with the foliation-preserving diffeomorphisms  $\text{Diff}_{\mathcal{F}}(M)$  of (3.2) (see [12, 13]). Since the Cotton tensor transforms covariantly under local conformal transformations of space, the potential term (3.5) is invariant under (3.7). At the special value of  $\lambda = 1/3$ , the kinetic term also becomes invariant under (3.7).

In  $z = 3$  gravity, the leading  $C^2$  term (3.5) in  $\mathcal{V}$  is of the same dimension as the kinetic term  $\sim K^2$ , and dominates the potential at short distances. However, the gauge symmetries of  $\text{Diff}_{\mathcal{F}}(M)$  allow a number of relevant terms in  $\mathcal{V}$ , which affect the dynamics at long distances. The theory flows in the infrared to lower values of  $z$ , and ultimately to  $z = 1$ . Such relevant terms in  $\mathcal{V}$  can be generated without violating detailed balance by adding two relevant terms to the three-dimensional Chern-Simons action  $W$ : the Ricci scalar  $R$  and the cosmological constant term. This turns  $W$  into the action of topologically massive gravity, and results in the modified potential

$$\mathcal{V} = \frac{\kappa^4}{16w^4} C_{ij} C^{ij} + \dots - \frac{c^2}{2\kappa^2} (R - 2\Lambda). \quad (3.8)$$

(The “...” in (3.8) stand for intermediate terms of fourth and fifth order in spatial derivatives.)

From the perspective of the  $z = 3$  UV fixed point,  $c$  and  $\Lambda$  are relevant coupling constants, of dimension two (in the units of inverse spatial length). The last two terms in (3.8) are those that appear in the potential  $\mathcal{V}$  of general relativity. At long distances, it is natural to redefine the time coordinate to reflect the  $z = 1$  scaling, by setting  $x^0 = ct$ . The theory in the infrared then closely resembles general relativity, with the effective Newton constant given by  $G_N = \kappa^2/(32\pi c)$ .

#### 4. The Spectral Dimension of Spacetimes with Anisotropic Scaling

In order to compare the behavior of the spectral dimension in the lattice CDT approach [11] with the analytic approach of gravity at a Lifshitz point, we must extend the definition of spectral dimension to smooth spacetimes with anisotropic scaling (3.1).

What is the appropriate diffusion process to consider? Recall first [11] that in the relativistic case of  $z = 1$ , the spectral dimension of the Minkowski spacetime is measured by first

rotating to imaginary time,  $t = -i\tau$ . On the resulting Euclidean space, the diffusion process is described by the probability density  $\rho$  of (2.2), governed by the diffusion equation

$$\frac{\partial}{\partial \sigma} \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \left( \frac{\partial^2}{\partial \tau^2} + \Delta \right) \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma), \quad (4.1)$$

where  $\Delta \equiv \partial_i \partial_i$  is the *spatial* Laplacian. This can be naturally generalized to the case with dynamical critical exponent  $z \neq 1$ . In the theories of gravity with anisotropic scaling, the dynamics along the time dimension stays qualitatively the same as in the  $z = 1$  case (*i.e.*, the Lagrangian is of the same order in time derivatives). It is the spatial dynamics that changes with the changing potential  $\mathcal{V}$ . This suggests that the natural diffusion process at general  $z$  is governed by the anisotropic diffusion equation,

$$\frac{\partial}{\partial \sigma} \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \left\{ \frac{\partial^2}{\partial \tau^2} + (-1)^{z+1} \Delta^z \right\} \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma). \quad (4.2)$$

Indeed, both terms in the operator on the right-hand-side of the equation scale the same way under the anisotropic rescaling of space and time (3.1). The relative sign  $(-1)^{z+1}$  in (4.2) is determined from the requirement of ellipticity of the diffusion operator. The formula is valid for integer values of  $z$ , but our results below can be analytically continued to any positive real value of  $z$ .

The anisotropic diffusion equation (4.2) is solved by

$$\rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \int \frac{d\omega d^D \mathbf{k}}{(2\pi)^{D+1}} e^{i\omega(\tau-\tau') + i\mathbf{x} \cdot (\mathbf{x} - \mathbf{x}')} e^{-\sigma(\omega^2 + |\mathbf{k}|^{2z})}. \quad (4.3)$$

In order to determine the spectral dimension, we only need  $\rho$  at the coincident initial and final spacetime points,  $\mathbf{x} = \mathbf{x}'$  and  $\tau = \tau'$ ,

$$\rho(\mathbf{x}, \tau; \mathbf{x}, \tau; \sigma) = \int \frac{d\omega d^D \mathbf{k}}{(2\pi)^{D+1}} e^{\sigma(\omega^2 + |\mathbf{k}|^{2z})} = \frac{C}{\sigma^{(1+D/z)/2}}, \quad (4.4)$$

with some nonzero constant  $C$ . Using (2.1), we finally obtain the spectral dimension of space-time with anisotropic scaling,

$$d_s \equiv -2 \frac{d \log P(\sigma)}{d \log \sigma} = 1 + \frac{D}{z}. \quad (4.5)$$

This implies the central result of this paper: In the 3+1 dimensional spacetime with  $z = 3$ , the spectral dimension (4.5) is equal to  $d_s = 2$ . Under the influence of the relevant deformations, the theory flows to  $z = 1$  in the infrared, reproducing the macroscopic value  $d_s = 4$  at long distances.

This result has been evaluated for a fixed classical spacetime geometry, described by

$$g_{ij} = \delta_{ij}, \quad N = 1, \quad N_i = 0. \quad (4.6)$$

Consequently, (4.5) represents the leading value for  $d_s$  in the semiclassical approximation. The notion of the spectral dimension can be generalized to the full quantum path integral of

the system, by defining the covariant generalization of the diffusion operator on an arbitrary curved geometry, and averaging the return probability over all configurations in the path integral. For theories which exhibit the flat spacetime (4.6) as a classical solution, (4.5) represents the leading term in the semiclassical evaluation of the spectral dimension. With quantum corrections small at large distances, the dependence of the spectral dimension on the spacetime scale is dominated by the change in the anisotropic scaling of the classical solution, from  $z = 3$  in the UV to  $z = 1$  in the IR.

An effective cosmological constant in the IR might preclude the flat geometry (4.6) from being a classical solution. If that happens, the spectral dimension will become sensitive at cosmological scales to the characteristic curvature of spacetime. However, such finite-size effects will not change the effective spectral dimension (4.5) at intermediate scales.

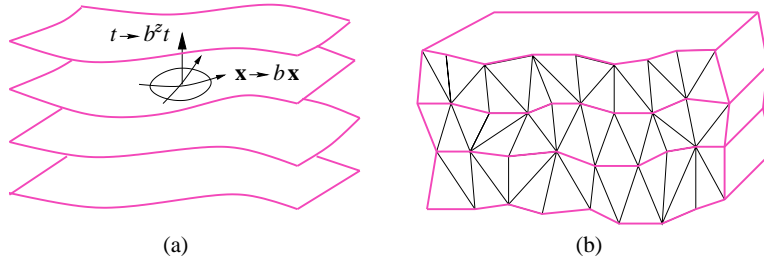
## 5. Discussion

We have extended the notion of spectral dimension to the continuum framework of quantum gravity with anisotropic scaling, and found that the behavior of spectral dimension matches qualitatively the lattice results obtained by Ambjørn *et al* [11] in the CDT approach. This raises the intriguing possibility that the continuum limit of the causal dynamical triangulations may belong to the same universality class as the anisotropic theory of gravity [13], flowing from the anisotropic scaling with  $z = 3$  in the UV to the relativistic value  $z = 1$  in the IR.

This possibility that the CDT lattice approach is effectively a lattice description of quantum field theory of gravity with anisotropic scaling presented in [12,13] is further supported by the symmetries imposed in the two frameworks. As reviewed above, theories of gravity with anisotropic scaling are invariant under foliation-preserving diffeomorphisms; the spacetime manifold is equipped with a preferred causal structure, compatible with anisotropic scaling (see Fig 1a). On the other hand, the novelty of the CDT approach to lattice gravity is that the sum is performed over lattice geometries with a preferred “causal structure” (Fig. 1b). Indeed, it is this extra condition on the discretizations which changes favorably the continuum limit, and prevents the collapse of the partition sum to a branched polymer phase. It is plausible that the continuum limit of the lattice sum automatically identifies a mechanism leading to its UV completion in the minimal way compatible with the preferred foliation, *i.e.*, in terms of a gravity theory with anisotropic scaling and  $z = 3$  at short distances.

The short-distance lattice value (2.5) of the spectral dimension is consistent within the margin of error with  $z = 3$ . However, the mean value reported is in fact closer to the spectral dimension of a smooth spacetime with  $z = 4$ . Theories with  $z = 4$  in  $3 + 1$  dimensions satisfying the detailed balance condition were discussed in [13]: They are constructed from the three-dimensional action  $W$  containing terms up to quadratic in the Ricci tensor. (Such models of three-dimensional gravity have recently been discussed in [26].) Reasons why gravity with  $z = 4$  might be desirable in  $3 + 1$  dimensions were discussed in [13].





**Figure 1:** (a) The preferred foliation by time slices in the continuum approach of gravity with anisotropic scaling; (b) a characteristic configuration in the causal dynamical triangulation approach.

Even though the main focus of this paper is on gravity in  $3 + 1$  dimensions, our result (4.5) for the spectral dimension of spacetime with anisotropic scaling is general, with possible applications to quantum gravity in other dimensions. For example, it is intriguing that the spectral dimensions (2.3) observed in the multicritical branched-polymer phases of discretized two-dimensional gravity can be reproduced by continuum theories in  $1 + 1$  dimensions with anisotropic scaling and the integer multicritical values of the dynamical exponent  $z = 2m - 1$ .

The spectral dimension also plays a prominent role in the thermal behavior of systems with anisotropic scaling. For example, simple scaling arguments show that the free energy of a system of free massless fields at the Lifshitz point with dynamical critical exponent  $z$  scales with temperature as  $F \sim T^{1+D/z} = T^{d_s}$ . Notably, when  $D = z$  (the critical dimension of gravity with anisotropic scaling), the behavior of the free energy  $F \sim T^2$  is the same as in relativistic CFT in  $1 + 1$  dimensions. This scaling has been seen before, by Atick and Witten [27] in their study of the ensemble of free strings, formally extrapolated into the regime above the Hagedorn temperature. An example of anisotropic gravity with  $z = 9$  in  $9 + 1$  dimensions can be obtained by following the logic of [13]: Starting with  $W \sim \int \omega_9$ , with  $\omega_9 = \Gamma \wedge (d\Gamma)^4 + \dots$  the Chern-Simons 9-form, and setting  $\mathcal{V} = (\delta W / \delta g_{ij})^2$  leads to a theory with detailed balance in  $9 + 1$  dimensions with  $z = 9$ , whose high-temperature behavior at the free-field fixed point matches the scaling found in [27] in ten-dimensional superstring theory formally extrapolated above the Hagedorn temperature.

In conclusion: We have demonstrated that even for smooth spacetime geometries, the spectral dimension of spacetime does not have to match its topological dimension. The discrepancy between the two can simply result from anisotropic scaling, compatible with a preferred causal structure of spacetime. This suggests an alternative interpretation of the dynamical reduction of spacetime at short distances [11] observed in the lattice approach to quantum gravity: This behavior does not have to indicate a change in the topological dimension of spacetime, or a foamy structure in which the four macroscopic dimensions result from coarse graining over topologically complicated two-dimensional geometries. Instead, the observed behavior of [11] can simply be explained by anisotropic scaling of space and time at short distances, keeping the topology of spacetime four-dimensional and its geometry smooth and topologically trivial.

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